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# Constructing Projective Algebras

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If $R$ is a commutative unitary ring, then not much is known about the projective objects in the category of $R$-algebras. It is easy to see that an $R$-algebra $B$ is projective if and only if $B$ is a retract of a polynomial ring over $R$ and it is well known that if $E$ is a projective $R$-module, then the symmetric algebra of $E$ is a projective $R$-algebra. The latter algebras are the "trivial" examples in this context and for a decade or so people have tried to both construct and understand nontrivial examples.

Recently, Yanik [7] and Connell and Wright [5], working independently, succeeded in extending a construction of Milnor for projective modules to projective algebras. The construction is quite general, but the proofs are long and involve proving an extension of the "Asanuma lifting lemma." Unaware of the above work, Greither gave an elementary proof of an important special case of the construction in his interesting paper [6]. The present authors upon reading [6], but also unaware of [5, 7], gave an elementary proof of a multivariable extension of Greither's single variable construction. Indeed, our paper is complementary to that of Greither in the sense that it represents out attempt to extend to several variables the results of [6] which are only proved in the case of a single variable. As ever, some extensions are possible and some seem impossible.

Thus, in the first section we state and prove multivariable versions of Greither's theorems. Specifically, we give a proof that the Milnor construction yields projective $R$-algebras in the case of many variables and in this case we also calculate the module of differentials of the resulting $R$-algebra.

In the second section we turn to the (concrete) problem of trying to determine when the construction can be used to give nontrivial examples. On this problem, the generality of the construction, as given in [5, 7] is of limited value. Greither showed persuasively that his construction was powerful in terms of giving examples. In fact, he was able to determine precisely the conditions on the base ring $R$ essential for the existence of nontrivial examples of projectives algebras "in a single variable." We present



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a sharply defined version of his results in Theorem 4 for the case of an integral domain.

We then try to determine the conditions on a domain under which the multivariable Milnor construction will yield only trivial examples. It turns out again that seminormality is a necessary condition and that complete normality is a sufficient condition; moreover, Prüfer domains yield only trivial examples if and only if valuation domains yield only trivial examples. Thus, we conclude by conjecturing that the Milnor construction, applied in case the base ring is a valuation domain, yields a polynomial ring.

## 1

In this section we shall give the construction referred to in the introduction. As noted there, it is an extension to several variables of the construction of Greither given in [6]. Specifically, we have the following setup.

$R \subseteq S$ are rings with $I$ a nonzero *common ideal* of $R$ and $S$. We have polynomials $\phi_1, ..., \phi_n \in S[Y_1, ..., Y_n]$, $Y_1, ..., Y_n$ indeterminates. Moreover, the images of $\phi_1, ..., \phi_n$ in $(S/I)[Y_1, ..., Y_n]$ determine an $(S/I)$-automorphism $\Phi$ of $(S/I)[Y_1, ..., Y_n]$; that is, $S[Y_1, ..., Y_n] = S[\phi_1, ..., \phi_n] + I[Y_1, ..., Y_n]$. Let $A_\Phi$ be the pullback of the diagram of $R$-algebras

$$\begin{tikzcd}
A_\Phi \arrow[r, dashed] \arrow[dd, dashed] & S[Y_1, ..., Y_n] \arrow[d] \\
 & (S/I)[Y_1, ..., Y_n] \arrow[d, "\Phi", "\wr"'] \\
(R/I)[Y_1, ..., Y_n] \arrow[r, hook] & (S/I)[Y_1, ..., Y_n].
\end{tikzcd}$$

Thus, $A_\Phi = \{(\bar{f}, g) \in (R/I)[Y_1, ..., Y_n] \oplus S[Y_1, ..., Y_n] \mid \Phi(\bar{f}) = \bar{g}\}$, where $\bar{}$ denotes coefficients modulo $I$.

Claim 1. *$A_\Phi$ can be identified with the $R$-algebra $R[\phi_1, ..., \phi_n] + I[Y_1, ..., Y_n]$.*

*Proof.* The map $(R/I)[Y_1, ..., Y_n] \hookrightarrow (S/I)[Y_1, ..., Y_n] \to^{\Phi} (S/I)[Y_1, ..., Y_n]$ is injective and so there is a one–one correspondence between $A_\Phi$ and $\{g \in S[Y_1, ..., Y_n] \mid \bar{g} = \Phi(\bar{f})$ for some $\bar{f} \in (R/I)[Y_1, ..., Y_n]\}$. Thus, $A_\Phi$ may be identified with the inverse image of $\Phi((R/I)[Y_1, ..., Y_n])$ under the map $S[Y_1, ..., Y_n] \to (S/I)[Y_1, ..., Y_n]$. But $\Phi((R/I)[Y_1, ..., Y_n])$ is equal to $(R/I)[\bar{\phi}_1, ..., \bar{\phi}_n]$ and this justifies the claim.

CLAIM 2. *The map $\tilde{\Phi}$ induced on $(S/I^2)[Y_1,..., Y_n]$ by $\Phi$ is an $(S/I^2)$-automorphism.*

*Proof.* Since $\Phi$ gives an automorphism of $(S/I)[Y_1,..., Y_n]$, $(S/I)[Y_1,..., Y_n] = (S/I)[\bar{\phi}_1,..., \bar{\phi}_n]$ and so $S[Y_1,..., Y_n] = S[\phi_1,..., \phi_n] + I[Y_1,..., Y_n]$. Hence, $I[Y_1,..., Y_n] = I[\phi_1,..., \phi_n] + I^2[Y_1,..., Y_n]$ and it follows that

$$S[Y_1,..., Y_n] = S[\phi_1,..., \phi_n] + I^2[Y_1,..., Y_n].$$

Therefore, $(S/I^2)[Y_1,..., Y_n] = (S/I^2)[\bar{\bar{\phi}}_1,..., \bar{\bar{\phi}}_n]$, where $\bar{\bar{\ }}$ denotes reducing coefficients mod $I^2$. But a surjective endomorphism of $(S/I^2)[Y_1,..., Y_n]$ is an automorphism [1, p. 315] and hence $\tilde{\Phi}$ is an $(S/I^2)$-automorphism of $(S/I^2)[Y_1,..., Y_n]$.

With all these preliminaries in hand we state our first result.

THEOREM 1. *The R-algebra $A_\Phi$ is a projective R-algebra.*

*Proof.* Let $\psi_1,..., \psi_n$ be polynomials in $S[Y_1,..., Y_n]$ which, when reduced modulo $I^2$, give the inverse of $\tilde{\Phi}$. Thus, there are polynomials $g_i$, $h_i \in I^2[Y_1,..., Y_n]$ so that

$$Y_i = \psi_i(\phi_1,..., \phi_n) + g_i,$$
$$Y_i = \phi_i(\psi_1,..., \psi_n) + h_i \qquad \text{for} \quad 1 \leqslant i \leqslant n.$$

We may write $g_i = \sum_{k=1}^{s} a_{ik} f_k$, where $a_{ik} \in I$ and $f_k \in I[Y_1,..., Y_n]$. We will show that $A_\Phi$ is a retract of $R[X_1,..., X_n, T_1,..., T_s]$ by producing maps

$$\rho: S[X_1,..., X_n, T_1,..., T_s] \to S[Y_1,..., Y_n]$$

and

$$\delta: S[Y_1,..., Y_n] \to S[X_1,..., X_n, T_1,..., T_s]$$

and proving that

(i) $\rho(R[X_1,..., X_n, T_1,..., T_s]) \subseteq A_\Phi$,

(ii) $\delta(A_\Phi) \subseteq R[X_1,..., X_n, T_1,..., T_s]$, and

(iii) $\rho \circ \delta = 1_{S[Y_1,...,Y_n]}$.

(Here, $X_1,..., X_n$, $T_1,..., T_s$ denote indeterminates.)

So, define $\rho$ to be the $S$-algebra map which is given by $X_i \mapsto \phi_i$ and $T_k \mapsto f_k$. It is clear that $\rho(R[X_1,..., X_n, T_1,..., T_s]) = R[\phi_1,..., \phi_n, f_1,..., f_s] \subseteq R[\phi_1,..., \phi_n] + I[Y_1,..., Y_n] = A_\Phi$.

Define $\delta$ to be the map determined by $Y_i \to \psi_i(X_1,..., X_n) + \sum_{k=1}^s a_{ik} T_k$. We must prove that $\delta(A_\Phi) \subseteq R[X_1,..., X_n, T_1,..., T_s]$. Now,

$$\begin{aligned}
&\delta(\phi_i(Y_1,..., Y_n)) \\
&\quad = \phi_i(\delta(Y_1),..., \delta(Y_n)) \\
&\quad = \phi_i \left(\psi_1(X_1,..., X_n) + \sum_{k=1}^{s} a_{1k} T_k,..., \psi_n(X_1,..., X_n) + \sum_{k=1}^{s} a_{nk} T_k\right) \\
&\quad = \phi_i(\psi_1(X_1,..., X_n),..., \psi_n(X_1,..., X_n)) \qquad \operatorname{mod} I[X_1,..., X_n, T_1,..., T_s] \\
&\quad = X_i - h_i \qquad \operatorname{mod} I[X_1,..., X_n, T_1,..., T_s] \\
&\quad = X_i \qquad \operatorname{mod} I[X_1,..., X_n, T_1,..., T_s].
\end{aligned}$$

Thus, $\delta(\phi_i) \in R[X_1,..., X_n, T_1,..., T_s]$ since $I$ is an ideal of both $R$ and $S$. Also, $\delta(I[Y_1,..., Y_n]) \subseteq I[X_1,..., X_n, T_1,..., T_s] \subseteq R[X_1,..., X_n, T_1,..., T_s]$.

Finally, we show that $\rho \circ \delta = 1_{S[Y_1,...,Y_n]}$, and to do this, it clearly suffices to prove it for $Y_1,..., Y_n$. But,

$$\begin{aligned}
\rho \circ \delta(Y_i) &= \rho \left(\psi_i(X_1,..., X_n) + \sum_{k=1}^{s} a_{ik} T_k\right) \\
&= \psi_i(\rho(X_1),..., \rho(X_n)) + \sum_{k=1}^{s} a_{ik} \rho(T_k) \\
&= \psi_i(\phi_1,..., \phi_n) + \sum_{k=1}^{s} a_{ik} f_k \\
&= \psi_i(\phi_1,..., \phi_n) + g_i \\
&= Y_i.
\end{aligned}$$

Before stating and proving our next result, we record a calculation we shall need which is similar to those in the preceding argument.

LEMMA 2. *Retaining the previous notation,* $A_\Phi \cdot I = I[Y_1,..., Y_n]$.

*Proof.* As above, we have that $S[Y_1,..., Y_n] = S[\phi_1,..., \phi_n] + I^2[Y_1,..., Y_n]$ and, keeping the same notation, $\psi_i(\phi_1,..., \phi_n) - Y_i \in I^2[Y_1,..., Y_n]$, while $\phi_1(\psi_1,..., \psi_n) - Y_i \in I^2[Y_1,..., Y_n]$.

Clearly, $A_\Phi \cdot I \subseteq I[Y_1,..., Y_n]$. Take a polynomial $F \in I[Y_1,..., Y_n]$. Then $F(\psi_1,..., \psi_n) \in I[Y_1,..., Y_n]$ and so $F(\psi_1(\phi_1,..., \phi_n),..., \psi_n(\phi_1,..., \phi_n)) \in I[\phi_1,..., \phi_n]$. Therefore,

$$\begin{aligned}
F = F(Y_1,..., Y_n) &= F(\psi_1(\phi_1,..., \phi_n),..., \psi_n(\phi_1,..., \phi_n)) \\
&\quad + \text{elements of } I^2[Y_1,..., Y_n]
\end{aligned}$$

belongs to

$$I[\phi_1,\ldots,\phi_n] + I^2[Y_1,\ldots,Y_n] = (R[\phi_1,\ldots,\phi_n] + I[Y_1,\ldots,Y_n])I = A_\Phi \cdot I.$$

In order to put the above construction to good use, it is often beneficial to know the module of Kähler differentials of $A_\Phi$ over $R$. Our next result provides this information.

PROPOSITION 3. *Let $A$ be the $A_\Phi$ of Theorem* 1. *Then*

(1) *The module of Kähler differentials of $A_\Phi$ over $R$, $\Omega_{A/R}$ is equal to $A(d\phi_1,\ldots,d\phi_n) + I[Y_1,\ldots,Y_n](dY_1,\ldots,dY_n)$.*

(2) *Moreover, $\Omega_{A/R}$ is a projective $A_\Phi$-module and is obtained by glueing together two free modules of rank $n$ over $(R/I)[Y_1,\ldots,Y_n]$ and $S[Y_1,\ldots,Y_n]$ via the Jacobian automorphism $(\partial\bar{\phi}_j/\partial Y_i)$ of $((S/I)[Y_1,\ldots,Y_n])^n$.*

*Proof.* We first show that $\Omega_{A/R}$ may be identified with its image under the natural map $j\colon \Omega_{A/R} \to \Omega_{S[Y_1,\ldots,Y_n]/S}$. Let $K$ be the total quotient ring of $A$. Consider the commutative diagram

$$\begin{tikzcd} \Omega_{A/R} \arrow[r, "i"] \arrow[dr, "j"'] & (\Omega_{A/R}) \otimes_R K \simeq \Omega_{K[Y_1,\ldots,Y_n]/K} \\ & \Omega_{S[Y_1,\ldots,Y_n]/S}. \arrow[u] \end{tikzcd}$$

Since $A$ is a retract of a polynomial ring $R[T_1,\ldots,T_m]$ over $R$, we see that $\Omega_{A/R}$ can be embedded in the free $R$-module $\Omega_{R[T_1,\ldots,T_m]/R}$. Hence, $i$ is injective and it follows that $j$ is also injective.

The module of differentials $\Omega_{S[Y_1,\ldots,Y_n]/S}$ is equal to $\coprod_{i=1}^n (S[Y_1,\ldots,Y_n])\,dY_i$. It is sometimes convenient to view $\Omega_{S[Y_1,\ldots,Y_n]/S}$ as $(S[Y_1,\ldots,Y_n])^n$ under the identification of

$$\begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \quad \text{with } \sum_{i=1}^n g_i\,dY_i \qquad \text{for} \quad g_i \in S[Y_1,\ldots,Y_n].$$

Now the image of $j$ is certainly equal to $AI(dY_1,\ldots,dY_n) + A(d\phi_1,\ldots,d\phi_n) = I[Y_1,\ldots,Y_n](dY_1,\ldots,dY_n) + A(d\phi_1,\ldots,d\phi_n)$ by Lemma 2.

As for the second assertion, recall that the $R$-algebra $A$ has been realized as the pullback of the diagram

$$\begin{tikzcd} & S[Y_1,\dots,Y_n] \arrow[d, "\sigma"] \\ (R/I)[Y_1,\dots,Y_n] \arrow[r, hook, "f"] & (S/I)[Y_1,\dots,Y_n] \end{tikzcd}$$

with $f(Y)=\bar{\phi}_i$ and $\sigma$ the canonical map. Let $M$ be the pullback of

$$\begin{tikzcd} & (S[Y_1,\dots,Y_n])^n \arrow[d, "\sigma"] \\ & ((S/I)[Y_1,\dots,Y_n])^n \\ ((R/I)[Y_1,\dots,Y_n])^n \arrow[r, hook, "f"] & ((S/I)[Y_1,\dots,Y_n])^n \arrow[u, "J"'] \end{tikzcd}$$

with $J$ the Jacobian $(\partial\bar{\phi}_j/\partial Y_i)_{ij}$. Since $f$ is injective, $M$ may be identified with $\sigma^{-1}\circ J\circ F((R/I)[Y])$, which is

$$I[Y_1,\dots,Y_n](S[Y_1,\dots,Y_n])^n+R[\phi_1,\dots,\phi_n]\left(\begin{bmatrix}\dfrac{\partial\phi_1}{\partial Y_1}\\ \vdots\\ \dfrac{\partial\phi_n}{\partial Y_n}\end{bmatrix},\dots,\begin{bmatrix}\dfrac{\partial\phi_n}{\partial Y_1}\\ \vdots\\ \dfrac{\partial\phi_n}{\partial Y_n}\end{bmatrix}\right),$$

but this equal to $I[Y_1,\dots,Y_n](dY_1,\dots,dY_n)+R[\phi_1,\dots,\phi_n](d\phi_1,\dots,d\phi_n)$. From the first part of the proof, we know that this last module is the $A$-submodule $\Omega_{A/R}$ of $\Omega_{S[Y_1,\dots,Y_n]/S}$.

Thus, $\Omega_{A/R}$ is the pullback of the above diagram of free modules and consequently [8, Theorem 2.1] is a projective $A_\Phi$-module.

## 2

In this section we turn to the problem of determining when the $A$'s constructed in the previous section can be used to give nontrivial examples of projective algebras. In the single variable case this was the thrust of the very interesting paper of Greither [6] and our next result could be viewed as an embellished summary of [6]. Greither was always dealing with reduced rings and often with reduced Noetherian rings $R$ having affine normalizations. The purpose of the finiteness assumptions was to insure that the seminormalization of $R$ was contained in the total quotient ring of $R$. We have chosen instead to assume that our rings are integral domains. This has the same effect, and in addition reduces Greither's notion of "algebra in one variable" to its essential ingredient, transcendence degree one over $R$. Thus, throughout this section $D$ and $S$ will be *integral domains*.

We note that in the setup of Section 1, $S$ must be contained in the quotient field $K$ of $D$. The reason is that $I$ is a common ideal of $D$ and $S$. Moreover, since $I[Y_1,..., Y_n] \subseteq A_\Phi$ and since the "quotient field" of $I$ is $K$, the quotient field of $A_\Phi$ is $K(Y_1,..., Y_n)$. Thus, the transcendence degree of $A_\Phi$ over $D$ is $n$.

We now state the aforementioned theorem of Greither, which is definitive in the one-dimensional case. Recall that $D$ is *seminormal* if, whenever $\alpha \in K$ with $\alpha^2, \alpha^3 \in D$, it follows that $\alpha \in D$ [4]. Recall also that a $D$-algebra $B$ is said to be *invertible* if there is a $D$-algebra $C$ such that $B \otimes_D C$ is a polynomial ring over $D$.

THEOREM 4. *Let $D$ be an integral domain with quotient field $K$. The following statements are equivalent.*

(a) *$D$ is seminormal.*

(b) *Each projective algebra of transcendence degree one over $D$ is a symmetric algebra.*

(c) *Each $A_\Phi$ in one variable is a symmetric algebra.*

(d) *If $D \subseteq S \subseteq K$ and if $I$ is a common ideal of $D$ and $S$, then the radical of $I$ in $S$ is contained in the conductor of $D$ in $S$.*

(e) *Each projective algebra of transcendence degree one over $D$ is invertible.*

*Proof.* That (a) is equivalent to (b) is the main theorem of [6] (stated for reduced rings). That (b) implies (c) is obvious and the proof of Corollary 3.6 of [6] shows that (c) implies (a).

(a) $\Leftrightarrow$ (d): We first recall from [4, Theorem 1] that seminormality of $D$ is equivalent to the following condition: For each $\alpha$ integral over $D$ but not in $D$, the conductor of $D$ in $D[\alpha]$ is a radical ideal of $D[\alpha]$. Thus, condition (d) is formally stronger than condition (a). For the converse, let $s \in S$ with $s^m \in I$. If $x \in S$, then $(xs)^m \in I$ since $I$ is a common ideal. If $xs \notin D$, then $xs$ is integral over $D$ and so, by the result quoted above, the conductor of $D$ in $D[xs]$ is a radical ideal of $D[xs]$. But $(xs)^m \in I$ and $IS = I \subseteq D$. Thus, $(xs)^m D[xs] \subseteq D$ and so $xs$ belongs to the radical of the conductor of $D$ in $D[xs]$. It follows that $xs \in D$ and therefore that $s$ belongs to the conductor of $D$ in $S$ as asserted.

(b) $\Rightarrow$ (e): This is well known and follows from the fact that symmetric algebra turns direct sum into tensor product.

(e) $\Rightarrow$ (a): This can be argued as in the proof of [6, Corollary 4.4]. Of course, one has to check that the argument of Corollary 4.4 remains valid when "reduced Noetherian ring with affine normalization" is replaced by "integral domain."

*Remark* 1. Conditions (a) through (c) are equivalent in case $R$ is reduced, but then one needs the notion of seminormality due to Swan [10].

*Remark* 2. Condition (d) is more natural in this context than it might seem and, in fact, the connection between conditions (c) and (d) is stronger than it first appears. Given a particular $D$, $S$, and $I$ as in the hypotheses of condition (d), but with the radical of $I$ not contained in the conductor of $D$ in $S$, we can use this condition to produce an $A_\Phi$ in one variable which is not a symmetric algebra. Indeed, if $x \in S$ with $x^m \in I$, but $x$ is not in the conductor, then choose an $s \in S$ with $xs \notin D$. The polynomial $\phi = Y_1 + xsY_1^2$ gives an automorphism of $(S/I)[Y_1]$, because $(xs)^m \in I$, but the resulting $A_\Phi$ is not a symmetric algebra by [6, Theorem 3.4].

*Remark* 3. At first glance one might hope that the above conditions might be equivalent to the condition: "All invertible $D$-algebras of transcendence degree one over $D$ are symmetric algebras." This is not the case and, in fact, as stated in [6], this condition is equivalent (when $D$ is an integral domain) to the condition referred to in [2] as $F$-closure. An integral domain $T$ with quotient field $L$ is called *$F$-closed* (or *steadfast*) if, whenever $\alpha \in L$ with $\alpha^2, \alpha^3 \in T$ and $n\alpha \in T$ for some positive integer $n$, it follows that $\alpha \in T$. Thus, $F$-closure is a weak form of seminormality. In particular, if $Q$ is the rational field and $X$ is an indeterminate, then $D = Q[X^2, X^3]$ is $F$-closed, but not seminormal. Hence, all invertible $D$-algebras of transcendence degree one over $D$ are symmetric algebras, but some $A_\Phi$'s are not invertible.

In trying to find a version of Theorem 4 which will be valid in several variables, one confronts the type of difficulty always present in such situations; the things one can do seem easy and the things one cannot seem impossible. For example, there is a multivariable extension of (c) $\Rightarrow$ (a).

PROPOSITION 5. *Let $D$ be a domain which is not seminormal. For each positive integer $n$, there are projective algebras $A_\Phi$ in $n$ variables which are not invertible.*

*Proof.* Using Theorem 4.3(b) of [6] there is a polynomial $\phi_1$ in $D[Y_1]$ such that the resulting pullback $A$ is not invertible. (Here again, it is necessary to replace "Noetherian having affine normalization" with "integral domain.") For $2 \leqslant i \leqslant n$, let $\phi_i = Y_i$ in $D[Y_1,..., Y_n]$. The resulting $A_\Phi$ is a projective $D$-algebra by Theorem 1, but

$$\begin{aligned} A_\Phi &= D[\phi_1, Y_2,..., Y_n] + I[Y_1,..., Y_n] \\ &= (D[\phi_1] + I[Y_1])[Y_2,..., Y_n]. \end{aligned}$$

Thus, $A_\Phi = A \otimes_D D[Y_2,..., Y_n]$ and $A_\Phi$ cannot be invertible because $A$ is not.

Now, another consequence of Theorem 4 is that in the transcendence degree one case, the $A_\Phi$'s in one variable are generic test algebras in the sense that if they are symmetric $D$-algebras, then all projective $D$-algebras are symmetric. Thus, it is natural to try to determine when the $A_\Phi$'s are symmetric algebras. Here, we are using "when" to mean that, given the data $D$, $S$, $I$, and $\Phi$, is $A_\Phi$ a symmetric $D$-algebra? Proposition 5 shows that some form of normality is required, but we have been unable to determine the exact form. Thus, our results are incomplete. They do, however, in our judgement point the way.

We begin with a result which shows that the $A_\Phi$'s are of no use when $D$ is a Noetherian normal domain.

PROPOSITION 6. *Let $D$ be an integral domain with quotient field $K$. If $S$ is an intermediate ring and if the conductor of $D$ in $S$ is nonzero, then $S$ is almost integral over $D$. Thus, if $D$ is completely normal, then $D = S$. In particular, if $D$ is a Noetherian normal domain* (*or more generally a Krull domain*), *the construction of Section* 1 *trivializes.*

*Proof.* Let $I$ be any nonzero common ideal of $D$ and $S$. Let $d$ be a nonzero element of $I$ and let $s \in S$. Then $s^m \in S$ and $ds^m \in I \subseteq D$ for all positive integers $m$—that is, $s$ is almost integral over $D$.

Another well-known class of normal domains is the class of Prüfer domains and for this class we can say the following.

PROPOSITION 7. *If each $A_\Phi$ over a valuation domain is a polynomial ring, then each $A_\Phi$ over a Prüfer domain is a symmetric algebra.*

*Proof.* Let $D$ be a Prüfer domain. Then an $A_\Phi$ over $D$, being a retract of a finite polynomial ring over $D$, is an affine $D$-algebra. Since $A_\Phi$ is an integral domain and since torsion-free $D$-modules are $D$-flat, $A_\Phi$ is finitely presented by [9, Corollary 3.4.7]. Thus, by [3], $A_\Phi$ is a symmetric algebra over $D$ if and only if $A_\Phi$ is locally a polynomial ring over $D_P$ for primes $P$ of $D$. Since $D_P$ is a valuation domain, we have only to see that $A_\Phi$'s localize well—that is, we must show that if $T$ is a multiplicatively closed subset of $D$, then $(A_\Phi)_T$ is constructed as in Section 1 from data over the ring $D_T$. But this is formal. More precisely, it follows at once from the following observation.

Let $R$ be a commutative ring and let $B$ be the pullback of the following diagram of $R$-algebras and $R$-algebra maps.

$$\begin{tikzcd} & R_2 \arrow[d, "f_2"] \\ R_1 \arrow[r, "f_1"'] & R' \end{tikzcd}$$

If $T$ is a multiplicatively closed subset of $R$, then $B_T$ is the pullback of the diagram

$$\begin{tikzcd} & (R_2)_T \arrow[d, "\hat{f}_2"] \\ (R_1)_T \arrow[r, "\hat{f}_1"'] & (R')_T \end{tikzcd}$$

of $R_T$-algebras.

Thus, to prove that all $A_\Phi$'s over a Prüfer domain are symmetric algebras, it suffices to do this for a valuation domain. This leads us to make the following conjecture.

CONJECTURE. *Let $V$ be a valuation domain with $P$ a prime ideal of $V$. Let $I$ be a common ideal of $V$ and $V_P$ and let $Y_1,\ldots, Y_n$ be indeterminates over $V$. Let $\phi_1,\ldots,\phi_n$ be polynomials in $V_P[Y_1,\ldots, Y_n]$ with $V_P[Y_1,\ldots, Y_n] = V_P[\phi_1,\ldots,\phi_n] + I[Y_1,\ldots, Y_n]$. Then $A_\Phi = V[\phi_1,\ldots,\phi_n] + I[Y_1,\ldots, Y_n]$ is a polynomial ring over $V$ (necessarily in $n$ variables).*

Of course, if this conjecture is true, then the net result is that the set of domains to which the construction can be effectively applied is rather limited.

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